Algorithm EKF_localization_known_correspondences($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, m$**):**

$$\theta = \mu_{t-1,\theta}$$

$$G_t = \begin{pmatrix} 1 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 1 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 1 \end{pmatrix}$$

$$V_t = \begin{pmatrix} \frac{-\sin\theta+\sin(\theta+\omega_t\Delta t)}{\omega_t} & \frac{v_t(\sin\theta-\sin(\theta+\omega_t\Delta t))}{\omega_t^2} + \frac{v_t\cos(\theta+\omega_t\Delta t)\Delta t}{\omega_t} \\ \frac{\cos\theta-\cos(\theta+\omega_t\Delta t)}{\omega_t} & -\frac{v_t(\cos\theta-\cos(\theta+\omega_t\Delta t))}{\omega_t^2} + \frac{v_t\sin(\theta+\omega_t\Delta t)\Delta t}{\omega_t} \\ 0 & \Delta t \end{pmatrix}$$

$$M_t = \begin{pmatrix} \alpha_1 v_t^2 + \alpha_2 \omega_t^2 & 0 \\ 0 & \alpha_3 v_t^2 + \alpha_4 \omega_t^2 \end{pmatrix}$$

$$\bar{\mu}_t = \mu_{t-1} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}$$

$$\bar{\Sigma}_t = G_t\ \Sigma_{t-1}\ G_t{}^T + V_t\ M_t\ V_t{}^T$$

$$Q_t = \begin{pmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\phi^2 & 0 \\ 0 & 0 & \sigma_s^2 \end{pmatrix}$$

for all observed features $z_t^i = (r_t^i\ \ \phi_t^i\ \ s_t^i)^T$ *do*

$$j = c_t^i$$

$$q = (m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2$$

$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(m_{j,y} - \bar{\mu}_{t,y}, m_{j,x} - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \\ m_{j,s} \end{pmatrix}$$

$$H_t^i = \begin{pmatrix} -\frac{m_{j,x}-\bar{\mu}_{t,x}}{\sqrt{q}} & -\frac{m_{j,y}-\bar{\mu}_{t,y}}{\sqrt{q}} & 0 \\ \frac{m_{j,y}-\bar{\mu}_{t,y}}{q} & -\frac{m_{j,x}-\bar{\mu}_{t,x}}{q} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_t^i = H_t^i\ \bar{\Sigma}_t\ [H_t^i]^T + Q_t$$

$$K_t^i = \bar{\Sigma}_t\ [H_t^i]^T [S_t^i]^{-1}$$

$$\bar{\mu}_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$$

$$\bar{\Sigma}_t = (I - K_t^i\ H_t^i)\ \bar{\Sigma}_t$$

endfor

$$\mu_t = \bar{\mu}_t$$

$$\Sigma_t = \bar{\Sigma}_t$$

$$p_{z_t} = \prod_i \det\left(2\pi S_t^i\right)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\ (z_t^i - \hat{z}_t^i)^T [S_t^i]^{-1} (z_t^i - \hat{z}_t^i)\right\}$$

return μ_t, Σ_t, p_{z_t}

Table 7.2 The extended Kalman filter (EKF) localization algorithm, formulated here for a feature-based map and a robot equipped with sensors for measuring range and bearing. This version assumes knowledge of the exact correspondences.

19 and 20 set the new pose estimate, followed by the computation of the measurement likelihood in line 21. In this algorithm, care has to be taken when computing the difference of two angles, since the result may be off by 2π.

7.4.3 Mathematical Derivation of EKF Localization

Prediction Step (Lines 3–7) The EKF localization algorithm uses the motion model defined in Equation (5.13). Let us briefly restate the definition:

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{\hat{v}_t}{\hat{\omega}_t} \sin\theta + \frac{\hat{v}_t}{\hat{\omega}_t} \sin(\theta + \hat{\omega}_t \Delta t) \\ \frac{\hat{v}_t}{\hat{\omega}_t} \cos\theta - \frac{\hat{v}_t}{\hat{\omega}_t} \cos(\theta + \hat{\omega}_t \Delta t) \\ \hat{\omega}_t \Delta t \end{pmatrix} \tag{7.4}$$

Here $x_{t-1} = (x \;\; y \;\; \theta)^T$ and $x_t = (x' \;\; y' \;\; \theta')^T$ are the state vectors at time $t-1$ and t, respectively. The true motion is described by a translational velocity, $\hat{v}_t$, and a rotational velocity, $\hat{\omega}_t$. As already stated in Equation (5.10), these velocities are generated by the motion control, $u_t = (v_t \;\; \omega_t)^T$, with additive Gaussian noise:

$$\begin{pmatrix} \hat{v}_t \\ \hat{\omega}_t \end{pmatrix} = \begin{pmatrix} v_t \\ \omega_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{\alpha_1 v_t^2 + \alpha_2 \omega_t^2} \\ \varepsilon_{\alpha_3 v_t^2 + \alpha_4 \omega_t^2} \end{pmatrix} = \begin{pmatrix} v_t \\ \omega_t \end{pmatrix} + \mathcal{N}(0, M_t) \tag{7.5}$$

We already know from Chapter 3 that EKF localization maintains a local posterior estimate of the state, represented by the mean μ_{t-1} and covariance Σ_{t-1}. We also recall that the "trick" of the EKF lies in linearizing the motion and measurement model. For that, we decompose the motion model into a noise-free part and a random noise component:

$$\underbrace{\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix}}_{x_t} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin\theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos\theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \; \Delta t \end{pmatrix}}_{g(u_t, x_{t-1})} + \mathcal{N}(0, R_t) \tag{7.6}$$

Equation (7.6) approximates Equation (7.4) by replacing the true motion $(\hat{v}_t \;\; \hat{\omega}_t)^T$ by the executed control $(v_t \;\; \omega_t)^T$, and capturing the motion noise in an additive Gaussian with zero mean. Thus the left term in Equation (7.6) treats the control as if it were the true motion of the robot. We recall from Chapter 3.3 that EKF linearization approximates the function g through a Taylor expansion:

$$g(u_t, x_{t-1}) \;\approx\; g(u_t, \mu_{t-1}) \;+\; G_t \,(x_{t-1} - \mu_{t-1}) \tag{7.7}$$

The function $g(u_t, \mu_{t-1})$ is simply obtained by replacing the exact state x_{t-1}—which we do not know—by our expectation μ_{t-1}—which we know. The Jacobian G_t is the derivative of the function g with respect to x_{t-1} evaluated at u_t and μ_{t-1}:

$$G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix} \tag{7.8}$$

Here $\mu_{t-1} = (\mu_{t-1,x} \quad \mu_{t-1,y} \quad \mu_{t-1,\theta})^T$ denotes the mean estimate factored into its individual three values, and $\frac{\partial x'}{\partial \mu_{t-1,x}}$ is short for the derivative of g along the x' dimension, taken with respect to x at μ_{t-1}. Calculating these derivatives from Equation (7.6) gives us the following matrix:

$$G_t = \begin{pmatrix} 1 & 0 & \frac{v_t}{\omega_t}(-\cos \mu_{t-1,\theta} + \cos(\mu_{t-1,\theta} + \omega_t \Delta t)) \\ 0 & 1 & \frac{v_t}{\omega_t}(-\sin \mu_{t-1,\theta} + \sin(\mu_{t-1,\theta} + \omega_t \Delta t)) \\ 0 & 0 & 1 \end{pmatrix} \tag{7.9}$$

To derive the covariance of the additional motion noise, $\mathcal{N}(0, R_t)$, we first determine the covariance matrix M_t of the noise in *control space*. This follows directly from the motion model in Equation (7.5):

$$M_t = \begin{pmatrix} \alpha_1 v_t^2 + \alpha_2 \omega_t^2 & 0 \\ 0 & \alpha_3 v_t^2 + \alpha_3 \omega_t^2 \end{pmatrix} \tag{7.10}$$

The motion model in (7.6) requires this motion noise to be mapped into *state space*. The transformation from control space to state space is performed by another linear approximation. The Jacobian needed for this approximation, denoted V_t, is the derivative of the motion function g with respect to the motion parameters, evaluated at u_t and μ_{t-1}:

$$\begin{aligned} V_t &= \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} \\ &= \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-\sin\theta + \sin(\theta + \omega_t \Delta t)}{\omega_t} & \frac{v_t(\sin\theta - \sin(\theta + \omega_t \Delta t))}{\omega_t^2} + \frac{v_t \cos(\theta + \omega_t \Delta t)\Delta t}{\omega_t} \\ \frac{\cos\theta - \cos(\theta + \omega_t \Delta t)}{\omega_t} & -\frac{v_t(\cos\theta - \cos(\theta + \omega_t \Delta t))}{\omega_t^2} + \frac{v_t \sin(\theta + \omega_t \Delta t)\Delta t}{\omega_t} \\ 0 & \Delta t \end{pmatrix} \end{aligned} \tag{7.11}$$

The multiplication $V_t \; M_t \; V_t^T$ then provides an approximate mapping between the motion noise in control space to the motion noise in state space.