4.3.3 Mathematical Derivation of the PF

To derive particle filters mathematically, it shall prove useful to think of particles as samples of state sequences

$$x_{0:t}^{[m]} = x_0^{[m]}, x_1^{[m]}, \ldots, x_t^{[m]} \tag{4.30}$$

It is easy to modify the algorithm accordingly: Simply append to the particle $x_t^{[m]}$ the sequence of state samples from which it was generated $x_{0:t-1}^{[m]}$. This particle filter calculates the posterior over all state sequences:

$$bel(x_{0:t}) = p(x_{0:t} \mid u_{1:t}, z_{1:t}) \tag{4.31}$$

instead of the belief $bel(x_t) = p(x_t \mid u_{1:t}, z_{1:t})$. Admittedly, the space over all state sequences is huge, and covering it with particles is usually not such a good idea. However, this shall not deter us here, as this definition serves only as the means to derive the particle filter algorithm in Table 4.3.

The posterior $bel(x_{0:t})$ is obtained analogously to the derivation of $bel(x_t)$ in Chapter 2.4.3. In particular, we have

$$\begin{aligned} & p(x_{0:t} \mid z_{1:t}, u_{1:t}) \\ \overset{\text{Bayes}}{=} \quad & \eta\, p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t})\, p(x_{0:t} \mid z_{1:t-1}, u_{1:t}) \\ \overset{\text{Markov}}{=} \quad & \eta\, p(z_t \mid x_t)\, p(x_{0:t} \mid z_{1:t-1}, u_{1:t}) \\ = \quad & \eta\, p(z_t \mid x_t)\, p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})\, p(x_{0:t-1} \mid z_{1:t-1}, u_{1:t}) \\ \overset{\text{Markov}}{=} \quad & \eta\, p(z_t \mid x_t)\, p(x_t \mid x_{t-1}, u_t)\, p(x_{0:t-1} \mid z_{1:t-1}, u_{1:t-1}) \end{aligned} \tag{4.32}$$

Notice the absence of integral signs in this derivation, which is the result of maintaining all states in the posterior, not just the most recent one as in Chapter 2.4.3.

The derivation is now carried out by induction. The initial condition is trivial to verify, assuming that our first particle set is obtained by sampling the prior $p(x_0)$. Let us assume that the particle set at time $t-1$ is distributed according to $bel(x_{0:t-1})$. For the m-th particle $x_{0:t-1}^{[m]}$ in this set, the sample $x_t^{[m]}$ generated in Step 4 of our algorithm is generated from the proposal distribution:

$$p(x_t \mid x_{t-1}, u_t)\, bel(x_{0:t-1}) = p(x_t \mid x_{t-1}, u_t)\, p(x_{0:t-1} \mid z_{1:t-1}, u_{1:t-1}) \tag{4.33}$$

with

$$w_t^{[m]} = \frac{\text{target distribution}}{\text{proposal distribution}} \tag{4.34}$$