The inner integral

$$\int V_{T-1}(b')\; p(b' \mid u, b, z)\; db' \tag{15.41}$$

contains only one non-zero term. This is the term where b' is the distribution calculated from b, u, and z using the Bayes filter. Let us call this distribution $B(b, u, z)$:

$$\begin{aligned} B(b,u,z)(x') &= p(x' \mid z, u, b) \\ &= \frac{p(z \mid x', u, b)\; p(x' \mid u, b)}{p(z \mid u, b)} \\ &= \frac{1}{p(z \mid u, b)}\; p(z \mid x') \int p(x' \mid u, b, x)\; p(x \mid u, b)\; dx \\ &= \frac{1}{p(z \mid u, b)}\; p(z \mid x') \int p(x' \mid u, x)\; b(x)\; dx \end{aligned} \tag{15.42}$$

The reader should recognize the familiar Bayes filter derivation that was extensively discussed in Chapter 2, this time with the normalizer made explicit.

We can now rewrite (15.40) as follows. Note that this expression no longer integrates over b'.

$$V_T(b) = \gamma \max_u \left[r(b,u) + \int V_{T-1}(B(b,u,z))\; p(z \mid u, b)\; dz \right] \tag{15.43}$$

This form is more convenient than the original one in (15.38), since it only requires integration over all possible measurements z, instead of all possible belief distributions b'. This transformation was used implicitly in the example above, where a new value function was obtained by mixing together finitely many piecewise linear functions.

Below, it will be convenient to split the maximization over actions from the integration. Hence, we notice that (15.43) can be rewritten as the following two equations:

$$V_T(b,u) = \gamma \left[r(b,u) + \int V_{T-1}(B(b,u,z))\; p(z \mid u, b)\; dz \right] \tag{15.44}$$

$$V_T(b) = \max_u V_T(b,u) \tag{15.45}$$

Here $V_T(b, u)$ is the horizon T-value function over the belief b, assuming that the immediate next action is u.

15.4.2 Value Function Representation

As in our example, we represent the value function by a maximum of a set of linear functions. We already discussed that any linear function over the