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The remaining statement, $\Sigma_{xx}^{-1} \mu_x = \bar{\xi}_x$, is obtained analogously, exploiting the fact that $\mu = \Omega^{-1} \xi$ (see Equation(3.73)) and the equality of the two expressions marked “(*)” above:

$$\begin{aligned} \begin{pmatrix} \Sigma_{xx}^{-1} \mu_x \\ 0 \end{pmatrix} &= \begin{pmatrix} \Sigma_{xx}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} = \begin{pmatrix} \Sigma_{xx}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \Omega^{-1} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} \\ &\stackrel{(*)}{=} \left[\Omega - \begin{pmatrix} 0 & \Omega_{xy} \Omega_{yy}^{-1} \\ 0 & 1 \end{pmatrix} \Omega \right] \Omega^{-1} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} \\ &= \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} - \begin{pmatrix} 0 & \Omega_{xy} \Omega_{yy}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = \begin{pmatrix} \bar{\xi}_x \\ 0 \end{pmatrix} \end{aligned}$$

Table 11.6 Lemma for marginalizing Gaussians in information form. The form of the **information** $\bar{\Omega}_{xx}$ in this lemma is also known as *Schur complement*.

Conditionals of a multivariate Gaussian. Let the probability distribution $p(x, y)$ over the random vectors x and y be a Gaussian represented in the information form

$$\Omega = \begin{pmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{yx} & \Omega_{yy} \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}$$

The conditional $p(x | y)$ is a Gaussian with information matrix Ω_{xx} and information vector $\xi_x - \Omega_{xy} y$.

Proof. The result follows trivially from the definition of a Gaussian in information form:

$$\begin{aligned} p(x | y) &= \eta \exp \left\{ -\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{yx} & \Omega_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} \right\} \\ &= \eta \exp \left\{ -\frac{1}{2} x^T \Omega_{xx} x - x^T \Omega_{xy} y - \frac{1}{2} y^T \Omega_{yy} y + x^T \xi_x + y^T \xi_y \right\} \\ &= \eta \exp \left\{ -\frac{1}{2} x^T \Omega_{xx} x + x^T (\xi_x - \Omega_{xy} y) - \underbrace{\frac{1}{2} y^T \Omega_{yy} y + y^T \xi_y}_{\text{const.}} \right\} \\ &= \eta \exp \left\{ -\frac{1}{2} x^T \Omega_{xx} x + x^T (\xi_x - \Omega_{xy} y) \right\} \end{aligned}$$

Table 11.7 Lemma for conditioning Gaussians in information form.